Three-term Machin-type formulae

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Introduction

The main theme in this talk is the Machin-type formula

$$\sum_{i=1}^{n} y_i \arctan \frac{1}{x_i} = \frac{r\pi}{4} \tag{1}$$

with integers $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ and $r \neq 0$.

The Machin's formula (Machin, 1706)

$$4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \frac{\pi}{4} \qquad (2)$$

is well known and have been used to calculate approximate values of π.

Analogous formulae:

$$\arctan\frac{1}{2}+\arctan\frac{1}{3}=\frac{\pi}{4}, \tag{3}$$

$$2\arctan\frac{1}{2}-\arctan\frac{1}{7}=\frac{\pi}{4} \tag{4}$$

and

$$2\arctan\frac{1}{3}+\arctan\frac{1}{7}=\frac{\pi}{4}. \tag{5}$$

(3)-(5) were attributed to Euler, Hutton and Hermann, respectively. But according to Tweddle, 1991, these formulae also seem to have been found by Machin.

Several three-term formulae also have been known (Simson, 1723 and Gauss, 1863 respectively):

$$8 \arctan \frac{1}{10} - \arctan \frac{1}{239} - 4 \arctan \frac{1}{515} = \frac{\pi}{4},$$

$$12 \arctan \frac{1}{18} + 8 \arctan \frac{1}{57} - 5 \arctan \frac{1}{239} = \frac{\pi}{4}.$$

Størmer, 1895: There exist only four two-terms formulae (2)-(5).

Størmer, 1896 gave a general criteria and 102 three-term formulae.

Størmer's criteria (revised)

A necessary and sufficient condition for given integers $x_1, x_2, \ldots, x_n > 1$ to have a Machin-type formula (1) is as follows:

$\exists s_{ij} (1 \leq i \leq n, 1 \leq j \leq n-1)$: integers,
$\exists \eta_1, \eta_2, \ldots, \eta_{n-1}$: Gaussian integers s.t.

$$\left[\frac{x_i + \sqrt{-1}}{x_i - \sqrt{-1}}\right] = \prod_{j=1}^{n-1} \left[\frac{\eta_j}{\bar{\eta}_j}\right]^{\pm s_{i,j}} \tag{6}$$

for $i = 1, 2, \ldots, n$.

Putting $m_i = \eta_i \bar{\eta}_i$, this condition can be reformulated as follows:

$\exists s_{i,j}(i,j = 1,2,\ldots,n)$: nonnegative integers with $0 \le s_{i,n} \le 1$ s.t.

$$x_i^2 + 1 = 2^{s_{i,n}} m_1^{s_{i,1}} m_2^{s_{i,2}} \cdots m_{n-1}^{s_{i,n-1}} \tag{7}$$

for $1 \le i \le n$ and $x_i \equiv \pm x_j \pmod{m_k}$ for two indices i,j with $x_i^2 + 1 \equiv x_j^2 + 1 \equiv 0 \pmod{m_k}$.

Størmer, 1896 questioned:

Are there infinitely many three-term formulae

$$\sum_{i=1}^{3} y_i \arctan \frac{1}{x_i} = \frac{r\pi}{4}, r \neq 0? \qquad (8)$$

The main result (Y,): There exist only finitely many integers $x_i, y_i (i = 1, 2, 3)$ and r with $x_1, x_2, x_3 > 1, \{x_1, x_2, x_3\} \neq \{2, 3, 7\}$ and $r \neq 0$ satisfying (8).

Furthermore,

I. If $x_i^2 + 1 \geq m_2$ for $i = 1, 2, 3$, then $m_1 < m_2 < 1.943109 \cdot 10^{48}, x_i < \exp(1.643 \cdot 10^{14})$ and $|y_i| < 1.092 \cdot 10^{24}$.

II. If $x_i^2 + 1 < m_2$ for some i, then $m_1 < 1.78731 \cdot 10^{76}, m_2 < \exp(1519318.88) < 6.4 \times 10^{659831}, x_i < \exp(2.367 \cdot 10^{23})$ and $|y_i| < 1.051 \cdot 10^{38}$.

The key tool: lower bound for linear forms in three logarithms.

This gives upper bounds for exponents k_i's and l_i's in terms of m_1, m_2,

Note: Depending on m_1, m_2, these upper bounds themselves do not give the desired finiteness.

However, provided that $r \neq 0$, we can prove the desired finiteness in the first case.

In the second case, we use a lower bound for a quantity of the form

$$y \arctan \frac{1}{x} - \frac{r\pi}{2},$$

which gives a linear form of two logarithms.

Notation and Preliminary lemmas

For any Gaussian integer η, we have an associate η' of η such that $-\pi/4 < \arg \eta' < \pi/4$ and therefore $-\pi/2 < \arg \eta'/\bar{\eta}' < \pi/2$.

Decompose $m_i = \eta_i \bar{\eta}_i$ in Gaussian integers so that $-\pi/4 < \arg \eta' < \pi/4$ and let $\xi_i = \eta_i / \bar{\eta}_i$.

Thus, $-\pi/2 < \arg \xi < \pi/2$.

For N composed of prime factors $\equiv 1 \pmod 4$,

$\widehat{\log} N := \log N$ if $N \geq 13$ and $\widehat{\log} 5 := 4 \arctan \frac{1}{2}$,

$\widetilde{\log} N := \max\{\log N, \frac{\log N}{2.648} + \max 4 \arg \frac{\eta}{\bar{\eta}}\}$,

$\gamma(N) = \widehat{\log} N / \log N, \delta(N) = \widetilde{\log} N / \log N$, where
$N = \eta\bar{\eta}$.

(1) is called **degenerate** if

$$\sum_{i\in S} y_i' \arctan \frac{1}{x_i} = \frac{r'\pi}{4} \tag{9}$$

for some proper subset S of $\{1,2,\ldots,n\}$, and integers $y_i'(i \in S)$ and r' which may be zero but not all zero

Lemma 1

In the three-term case, a degenerate cases occurs only if $\{x_1, x_2, x_3\} = \{2, 3, 7\}$.

This lemma follows from Størmer's result for two-terms formulas.

Lemma 2

The equation $x^2 + 1 = 2^e y^n$ with $x > 0, n > 2$ has only one integral solution $(x, e, y, n) = (239, 1, 13, 4)$.

$e = 0$: M. Lebesgue, 1850.
n odd, $e = 1$: Størmer, 1897.
$n = 4$: Ljunggren, 1942 (easier proofs by Wolskill, 1989 and Steiner and Tzanakis, 1991).

The key tool

A lower bound for linear forms of three logarithms

Results in Mignotte's *a kit on linear forms in three logarithms* are rather technical but still worthful to use for the purpose of improving our upper bounds.

$\alpha_1, \alpha_2, \alpha_3$: Gaussian rationals with $\alpha_i \neq 1, |\alpha_i| = 1$,

α_i's are multiplicatively independent or

two of these numbers are multiplicatively independent and the third one is a root of unity, i.e. -1 or $\pm\sqrt{-1}$.

b_1, b_2, b_3: three coprime positive rational integers and

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1 - b_3 \log \alpha_3, \tag{10}$$

where the logarithm of each α_i can be arbitrarily determined as long as

$$b_2 \left|\log \alpha_2\right| = b_1 \left|\log \alpha_1\right| + b_3 \left|\log \alpha_3\right| \pm \left|\Lambda\right|. \tag{11}$$

$d_1 = \mathrm{gcd}(b_1, b_2), d_2 = \mathrm{gcd}(b_2, b_3), b_2 = d_1 b_2' = d_3 b_2''$.

$w_i = |\log \alpha_i| = |\arg \alpha_i|$ for each $i = 1, 2, 3$,

a_1, a_2, a_3: real numbers such that
$a_i \geq \max\{4, 5.296 w_i + 2h(\alpha_i)\}$,

$\Omega = \max\{a_1 a_2 a_3, 100\}$.

Furthermore, put

$$b' = \left(\frac{b'_1}{a_2} + \frac{b'_2}{a_1}\right)\left(\frac{b''_3}{a_2} + \frac{b''_2}{a_3}\right) \qquad (12)$$

and $\log B = \max\{0.882 + \log b', 10\}$.

Then, either one of the following holds.

A. The estimate

$$\log|\Lambda| > -790.95\Omega\log^2 B \qquad (13)$$

holds.

B. There exist two nonzero rational integers r_0 and s_0 such that $r_0 b_2 = s_0 b_1$ with $|r_0| \leq 5.61 a_2 \log^{1/3} B$ and $|s_0| \leq 5.61 a_1 \log^{1/3} B$.

C. There exist four rational integers r_1, s_1, t_1 and t_2 with $r_1 s_1 \neq 0$ such that

$$\begin{aligned} &(t_1 b_1 + r_1 b_3) s_1 = r_1 b_2 t_2, \\ &\mathrm{gcd}(r_1, t_1) = \mathrm{gcd}(s_1, t_2) = 1 \end{aligned} \tag{14}$$

and

$$\begin{aligned} &(|s_1 t_1|, |r_1 t_2|, |r_1 s_1|) \\ &\leq 5.61 \delta \log^{1/3} B(a_1, a_2, a_3), \end{aligned} \tag{15}$$

where $\delta = \mathrm{gcd}(r_1, s_1)$. Moreover, when $t_1 = 0$ we can take $r_1 = 1$ and then $t_2 = 0$ we can take $s_1 = 1$.

This result is nonsymmetric for three logarithms and, in order to make each b_i positive, we should arrange the order of logarithms. Thus, the application of this result requires a fair amount of computations with many branches of cases.

We note that may assume that $\log\alpha_2 = \pi\sqrt{-1}/2$ or $\log\alpha_3 = \pi\sqrt{-1}/2$ if one of α_i's is a root of unity.

Upper bounds for the sizes of exponents

Now we shall obtain upper bounds for the size of a solution of

$$x^2 + 1 = 2^e m_1^{e_1} m_2^{e_2} \tag{16}$$

for given integers m_1, m_2.

If $e_1 = 0$, then $e_2 \in \{1, 2, 4\}$ by Lemma 2.

Hence, we may assume that $e_1 \neq 0$ and we can put $e_2 \log m_2 = \kappa e_1 \log m_1$. Thus, $e_1 \log m_1 + e_2 \log m_2 = (1 + \kappa) e_1 \log m_1$.

Moreover, put $\beta = \frac{\pi(1+\kappa)}{2} + 2 \max\{1, \kappa\}$.

We present only the case C with $\log \alpha_2 = \pi\sqrt{-1}/2$, which eventually gives the worst estimate for our main result:

$$
\begin{aligned}
&(1+\kappa)e_1 \log m_1 \\
<&60064 g(m_1, m_2) \log m_1 \log^2 m_2 \\
&\times \left(\log C_{4,1} + \frac{\log\log m_1 + 3\log\log m_2}{2}\right)^{8/3},
\end{aligned}
\tag{17}
$$

where

$$\begin{aligned} f(m_1, m_2) =& 1 + \frac{2.648\pi(\widehat{\log} m_1 + \log m_2)}{\widehat{\log} m_1 \log m_2}, \\ g(m_1, m_2) =& f(m_1, m_2)\gamma(m_1)\delta(m_2) \end{aligned}$$

and

$$C_{4,1} < 13262 g(m_1, m_2) \left(\frac{1}{5.296\pi} + \frac{1}{\log m_1} \right).$$

Let

$$\Lambda = \log \frac{x+\sqrt{-1}}{x-\sqrt{-1}} = \pm e_1 \log \xi_1 \pm e_2 \log \xi_2 \pm \frac{r\pi}{2}. \tag{18}$$

Provided that $m_1^{e_1} m_2^{e_2} > 10^{20}$,

$$\begin{aligned} \log|\Lambda| &< -\log x \\ &< -\frac{(1+\kappa)e_1 \log m_1}{2} + 10^{-9}. \end{aligned} \tag{19}$$

In the case $\log \alpha_2 = \pi i/2$, we can set:

$\alpha_i = \xi_{j_i}$,

$b_i = \left| e_{j_i} \right| / \gcd(e_1, e_2, r)$ for $i = 1, 3$, where $(j_1, j_3) = (1, 2)$ or $(2, 1)$.

$a_i = \widehat{\log}\, m_{j_i} + 5.296\pi\theta_{j_i} (i = 1, 3)$ and $a_2 = 2.648\pi$.

Reduction into two logarithms.

We put $r_1 = \delta r_0, s_1 = \delta s_0$, which immediately yields that $\gcd(r_0, s_0) = 1$.

We can see that $r_0 \mid b_1$ and $s_0 \mid b_2$.

Now put $b_1 = r_0 u_1, b_2 = s_0 u_2$. Dividing (14) by $r_0 s_0$, we have

$$t_1 u_1 + t_2 u_2 + \delta b_3 = 0. \tag{20}$$

Now we obtain

$$\delta\Lambda = u_2 \log \alpha_6 - u_1 \log \alpha_7, \qquad (21)$$

where

$$\alpha_6 = \alpha_2^{s_1}\alpha_3^{t_2}, \alpha_7 = \alpha_1^{r_1}\alpha_3^{-t_1}. \qquad (22)$$

Moreover,

$$\begin{aligned} |s_0 t_1| &\leq 5.61 a_1 \log^{1/3} B, \\ |r_0 t_2| &\leq 5.61 a_2 \log^{1/3} B, \\ |\delta r_0 s_0| &\leq 5.61 a_3 \log^{1/3} B. \end{aligned} \tag{23}$$

We take

$$a_i = \max\{h(\alpha_i), |\log \alpha_i|\}\,(i = 6, 7),$$
$$b'' = \frac{|u_1|}{a_6} + \frac{|u_2|}{a_7} \le \frac{b_1}{|s_0|\,a_6} + \frac{b_2}{|s_0|\,a_7}.$$

Then, Corollaire 1 of (Laurent, Mignotte and Nesterenko, 1995) gives

$$\log|\delta\Lambda| \ge -30.9\max\{\log^2 b'', 441\}a_6 a_7. \quad (24)$$

After some amount of calculation, we have

$$\frac{(1+\alpha)e_1 \log m_1}{2} < 6630 g(m_1, m_2) \log m_1 \log^2 m_2 \times \log^{2/3}\left(c_1^{1/2} e_1 \sqrt{\frac{\log m_1}{\log m_2}}\right) \log^2 b_0'', \tag{25}$$

where $b_0'' = \beta e_1 / \log m_2$ and c_1 denotes some constant such that

$$
\begin{aligned}
&\frac{1}{\log m_1}\left(\frac{1}{\log m_1} + \frac{1}{2.648\pi}\right) \\
&< \frac{c_1}{(1+\alpha)^2} \qquad (26) \\
&\quad < \left(\frac{1}{\log m_1} + \frac{1}{5.296\pi}\right)^2 .
\end{aligned}
$$

(25) gives (17).

We can prove for other cases in similar ways.

Outline of the proof of the main result

Assume

$$\sum_{i=1}^{3} y_i \arctan \frac{1}{x_i} = \frac{r\pi}{4} \tag{27}$$

with $x_1, x_2, x_3 > 1, r \neq 0$ and let

$$x_i^2 + 1 = 2^{v_i} m_1^{k_i} m_2^{l_i}. \tag{28}$$

We note that

$y_1 = \pm k_2 l_3 \pm k_3 l_2,$

$y_2 = \pm k_3 l_1 \pm k_1 l_3,$

$y_3 = \pm k_1 l_2 \pm k_2 l_1$

with appropriate choices of signs.

Let $K = \max k_i$ and $L = \max l_i$.

We have two cases: I. $x_1^2 + 1 \geq m_2$ and II. $x_1^2 + 1 < m_2$.

Case I.

In this case, $x_i \geq \sqrt{m_2 - 1}$ for $i = 1, 2, 3$.

We may assume that a) $l_1 l_2 l_3 > 0, x_2, x_3 > m_2/2$ or b) $l_1 = 0, l_2 l_3 > 0, x_3 > m_2/2$.

From (27) with $r \neq 0$,

$$\frac{|y_1| + |y_2|}{\sqrt{m_2 - 1}} + \frac{2\,|y_3|}{m_2} > \frac{\pi}{4}.$$

Since $|y_1| \leq k_2 l_3 + k_3 l_2 \leq 2KL$ and so on, we have $m_2 < (4(2 + 10^{-8})KL/\pi)^2 < 6.49(KL)^2$.

Combining with our upper bounds for exponents, we have
$m_1 < m_2 < 1.943109 \cdot 10^{48}$,
$|y_i| < 2KL < 1.092 \cdot 10^{24}$
and $\log x_i < k_i \log m_1 + l_i \log m_2 < 2KL \log m_2 < 1.643 \cdot 10^{14}$, that is, $x_i < \exp(1.643 \cdot 10^{14})$. This shows the Theorem in Case I.

Case II.

We may assume that $l_1 = 0, x_1^2 + 1 < m_2, l_2 l_3 > 0, x_3 > m_2/2$.

Now we clearly have $y_2 = \pm l_3 k_1, y_3 = \pm l_2 k_1$ and therefore

$$\left| \frac{r\pi}{4} - y_1 \arctan \frac{1}{x_1} \right| < \frac{(1 + 10^{-8}) k_1 (l_2 + l_3)}{m_2^{1/2}}, \tag{29}$$

with $|y_1| \le k_2 l_3 + l_3 k_2 < 2KL$.

Let

$$\Lambda_1 = 2y_1 \log \frac{x_1 + \sqrt{-1}}{x_1 - \sqrt{-1}} - r\pi\sqrt{-1}. \tag{30}$$

Then (27) gives

$$|\Lambda_1| < \frac{4(1 + 10^{-8})k_1 L}{m_2^{1/2}}, \tag{31}$$

while Théorème 3 of (LMN, 1995) gives that

$$-\log|\Lambda_1| < 8.87aH^2, \tag{32}$$

where

$$
\begin{aligned}
a &= \max\left\{20, 10.98\widehat{\log}\, m_1 + \frac{\log m_1}{2}\right\}, \\
H &= \max\left\{17, 2.38 + \log\left(\frac{r}{2a} + \frac{2y_1}{68.9}\right)\right\}.
\end{aligned}
$$

If $m_2 > e^{175}$, then (32) gives

$$\log m_2 < 155.77(10.98\gamma(m_1)+0.58)\log m_1 \log\log m_1. \tag{33}$$

Recalling (29), we have

$$\begin{aligned} \frac{2KL}{\sqrt{m_1-1}} &> \frac{\pi}{4} - \frac{2k_1L}{\sqrt{m_2-1}} \\ &> \frac{\pi(1-10^{-8})}{4}. \end{aligned} \tag{34}$$

Combining this with (33) and our upper bounds for exponents, we have
$m_1 < 2.9526 \cdot 10^{76}$,
$m_2 < \exp(1639526.95) < 3.2 \times 10^{712037}$,
$\log x_i = t_i \log m_1 + u_i \log m_2 < 2.6475 \cdot 10^{23}$,
that is, $x_i < \exp(2.6475 \cdot 10^{23})$,
and $y_i \leq 2TU < 1.281 \cdot 10^{38}$. This completes the proof of the Theorem.

Background

(7) can be seen as a special case of the generalized Ramanujan-Nagell equation

$$x^2 + Ax + B = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}, \qquad (35)$$

where A and B are given integers with $A^2 - 4B \neq 0$ and $p_1, p_2, \ldots, p_n$ are given primes.

Evertse, 1984: (35) has at most $3 \cdot 7^{4n+6}$ solutions.

In the case $n = 2$, Y., 2018 reduced Evertse's bound $3 \cdot 7^{14}$ to 63.

Our result does not give an upper bound for numbers of solutions

$$x^2 + 1 = 2^s p_1^k p_2^l \qquad (36)$$

since the case $r = 0$ is not considered.

Indeed, Størmer, 1896 implicitly pointed out that, if $x^2 + 1 = ay$, then

$$\begin{aligned} &\arctan \frac{1}{az - x} - \arctan \frac{1}{az + a - x} \\ &\quad = \arctan \frac{1}{az(z+1) - (2z+1)x + y}. \end{aligned} \tag{37}$$

Størmer, 1897: (36) has at most one solution with each fixed combination of parities of s_i, k_i, l_i with zero and nonzero-even distinguished.

Although there exist 18 combinations

$$(0 \mid 1, 0 \mid 1 \mid 2, 0 \mid 1 \mid 2),$$

all-even combinations can clearly be excluded and therefore (36) has at most 14 solutions totally.

Størmer, 1896 also questioned: Is there any further three-term formula? Up to now, the only known other nontrivial (i.e. not derived from (3)-(5)) three-term formulae are

$$5\arctan\frac{1}{2}+2\arctan\frac{1}{53}+\arctan\frac{1}{4443}=\frac{3\pi}{4},$$
$$5\arctan\frac{1}{3}-2\arctan\frac{1}{53}-\arctan\frac{1}{4443}=\frac{\pi}{2},$$
$$5\arctan\frac{1}{7}+4\arctan\frac{1}{53}+2\arctan\frac{1}{4443}=\frac{\pi}{4}.$$

Maurice Mignotte, *A kit on linear forms in three logarithms*, avilable from Y. Bugeaud's web page:
http://irma.math.unistra.fr/ bugeaud/travaux/kit.pdf

Carl Størmer, Sur l'application de la théorie des nombres entiers complexes a la solution en nombres rationnels $x_1, x_2, \ldots, x_n, c_1, c_2, \ldots, c_n, k$ de l'équation: $c_1$arc tg$x_1 + c_2$arc tg$x_2 + \cdots + c_n$arc tg$x_n = k\frac{\pi}{4}$, *Arch. Math. Naturv.* **19** (1896), Nr. 3, 96 pages.

Michael Roby Wetherfield and Hwang Chien-lih, Computing π, lists of Machin-type (inverse cotangent) identities for $\pi/4$,
http://www.machination.eclipse.co.uk/index.html

MANY THANKS
FOR YOUR ATTENTION

To be continued...?

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