Machin-type formulae

$$k_1 \arctan \frac{1}{x_1} + k_2 \arctan \frac{1}{x_2} + k_3 \arctan \frac{1}{x_3} = \frac{r\pi}{4} \text{ with}$$

$$2 \leq x_1 \leq 9$$

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Introduction

The Machin's formula (Machin, 1706)

$$4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \frac{\pi}{4} \tag{1}$$

is well known and have been used to calculate approximate values of π.

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Analogous formulae

$$\arctan\frac{1}{2}+\arctan\frac{1}{3}=\frac{\pi}{4}, \tag{2}$$

$$2\arctan\frac{1}{2}-\arctan\frac{1}{7}=\frac{\pi}{4}, \tag{3}$$

and

$$2\arctan\frac{1}{3}+\arctan\frac{1}{7}=\frac{\pi}{4}. \tag{4}$$

(2)-(4) were attributed to Euler, Hutton and Hermann, respectively. But according to Tweddle, 1991, these formulae also seem to have been found by Machin.

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Several three-term formulae also have been known. For example,

$$8\arctan\frac{1}{10}-\arctan\frac{1}{239}-4\arctan\frac{1}{515}=\frac{\pi}{4},$$

$$12\arctan\frac{1}{18}+8\arctan\frac{1}{57}-5\arctan\frac{1}{239}=\frac{\pi}{4}.$$

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Two-term Machin-type formulae (Størmer, 1895)

There exist only four two-term formulae (1)-(4).

A problem

For given $n \geq 3$, determine all n-term Machin-type formulae

$$k_1 \arctan \frac{1}{x_1} + k_2 \arctan \frac{1}{x_2} + \cdots + k_n \arctan \frac{1}{x_n} = \frac{r\pi}{4} \tag{5}$$

with $k_1, \ldots, k_n$, $x_1, \ldots, x_n$, and r integers with $x_1, \ldots, x_n \geq 2$ and $r \neq 0$.

This problem is unsolved even for $n = 3$.

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This problem is unsolved even for $n = 3$.

Størmer, 1896 gave a general criteria and 102 three-term formulae.

Størmer's criteria (revised)

A necessary and sufficient condition for given integers $x_1, x_2, \ldots, x_n \geq 2$ to have a Machin-type formula (5) is as follows:

$\exists s_{ij} (1 \leq i \leq n, 1 \leq j \leq n-1)$: integers,

$\exists \eta_1, \eta_2, \ldots, \eta_{n-1}$: Gaussian integers s.t.

$$\left[\frac{x_i + \sqrt{-1}}{x_i - \sqrt{-1}}\right] = \prod_{j=1}^{n-1} \left[\frac{\eta_j}{\bar{\eta}_j}\right]^{\pm s_{i,j}} \tag{6}$$

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Now we would like to confine our interest to three-type formulae

$$k_1 \arctan \frac{1}{x_1} + k_2 \arctan \frac{1}{x_2} + k_3 \arctan \frac{1}{x_3} = \frac{r\pi}{4} \tag{7}$$

in integers $k_1, k_2, k_3, x_1, x_2, x_3$, and r with $r \neq 0$ and $x_1, x_2, x_3 \geq 2$ and $m_j = \eta_j \bar{\eta}_j$ for $j = 1, 2$ with $m_1 < m_2$.

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Other than 102 formulae, only three nontrivial formulae are known:

$$5\arctan\frac{1}{2} + 2\arctan\frac{1}{53} + \arctan\frac{1}{4443} = \frac{3\pi}{4},$$

$$5\arctan\frac{1}{3} - 2\arctan\frac{1}{53} - \arctan\frac{1}{4443} = \frac{\pi}{2},$$

and

$$5\arctan\frac{1}{7} + 4\arctan\frac{1}{53} + 2\arctan\frac{1}{4443} = \frac{\pi}{4}.$$

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Previously, the speaker proved that:

Y., 2018, arXiv: 1811.09273

There exist only finitely many integers $x_i, k_i (i = 1, 2, 3)$ and r with $x_1, x_2, x_3 \geq 2$, $\{x_1, x_2, x_3\} \neq \{2, 3, 7\}$, $\gcd(k_1, k_2, k_3) = 1$, and $r \neq 0$ satisfying (7).

Y., 2018, revised in 2025, arXiv: 1811.09273

Furthermore,

I. If $m_1 m_2 \geq 10^{10}$ and $x_i^2 + 1 \geq m_2$ for $i = 1, 2, 3$, then $m_1 < m_2 < 1.342 \times 10^{34}$, $\log x_i < 83801148333$, and $|k_i| < 9.152 \cdot 10^{16}$.

II. If $m_1 m_2 \geq 10^{10}$ and m_2 does not divide $x_i^2 + 1$ for some i, then $m_1 < 2.531 \times 10^{24}$, $\log m_2 < 294622$, $\log x_i < 5.054 \times 10^{12}$, and $|y_i| < 1.312 \times 10^{19}$. Moreover, $\log m_2 < 40000$ if additionally $m_1 \geq 9.134 \times 10^{22}$.

III. If $m_1 m_2 < 10^{10}$, then $\log x_i < 8813999998$ and $|y_i| < 4.508 \times 10^{18}$.

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The speaker also proved:

Lemma 1, Y., 2018, revised in 2025 (continued)

If $m_1 = 5, 13, 17, 37$, and 41, then
$\log m_2 < 115315, 116234, 110031, 108986$, and 118023 respectively.
Moreover, if $53 \le m_1 < 5000$, then $\log m_2 < 138880$.

Proofs of these results involved a new lowerr bound for linear forms of three logarithms by Mignotte, Voutier, and Laurent and a new lower bound for linear forms of one logarithm and πi by the speaker (arXiv: 1906.00419).

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Main results

Theorem 1 (Y.)

(7) has no new solution with $x_1 \in \{2, 3\}$, $\{x_1, x_2, x_3\} \neq \{2, 3, 7\}$. (Known solutions up to 2023 have been given by Wetherfield and Hwang)

Note

This does not imply that (7) has no new solution with $x_1 = 7$. Although $\arctan(1/2)$ and $\arctan(1/7)$ are linearly dependent, if we replace (7) with $x_1 = 7$ and $r \neq 0$ by $x_1 = 2$, then r may become zero.

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Theorem 2 (Y.)

If we have (7) with $4 \le x_1 \le 9$, $\{x_1, x_2, x_3\} \ne \{2, 3, 7\}$, $\gcd(k_1, k_2, k_3) = 1$, and $m_1 m_2 > 10^{10}$, then x_2 and $|k_1|$ are bounded by constants given in the table below.

Table: Constants

x_1	m_1	$x_2 <$	$\lvert k_1 \rvert \le$
4	17	5.05×10^{16}	237902654
5	13	1.23×10^{17}	467372721
6	37	1.81×10^{16}	171102423
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Proof of main results

We begin by a consequene of Størmer's criteria:

If we have (7) in integers $k_1, k_2, k_3, x_1, x_2, x_3$, and r with $r \neq 0$ and $x_1, x_2, x_3 \geq 2$, then, putting $m_j = \eta_j \bar{\eta}_j$ for $j = 1, 2$ with $m_1 < m_2$, we have

$$x_i^2 + 1 = 2^{s_{i,0}} m_1^{s_{i,1}} m_2^{s_{i,2}}$$

with $s_{i,j}$ nonnegative integers for $i = 1, 2, 3$ and $j = 0, 1, 2$. Moreover, $s_{i,0} \in \{0, 1\}$ and we can take

$$k_i = \pm s_{i+1,1} s_{i+2,2} \pm s_{i+2,1} s_{i+1,2}$$

with an appropriate combination of signs for $i = 1, 2, 3$ (we take $s_{4,j} = s_{1,j}$ and $s_{5,j} = s_{2,j}$ for $j = 1, 2$).

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So that, we are led to exponential diophantine equations

$$x^2 + 1 = m_1^{e_1} m_2^{e_2}$$

and

$$x^2 + 1 = 2m_1^{e_1} m_2^{e_2}.$$

In Y. 2018/2025, the speaker proved that

Bounds for solutions of exponential diophantine equations

If $x^2 + 1 = m_1^{e_1} m_2^{e_2}$ or $2m_1^{e_1} m_2^{e_2}$ with $m_1 \in \{5, 13, 17, 37, 41, 65\}$, then, for $m_1 m_2 \geq 10^{10}$, we have

$$e_1 \log m_1 + e_2 \log m_2 \leq A_4 (\log m_1 \log m_2) \log^2 (A_5 \log m_1)$$

and otherwise

$$e_1 \log m_1 + e_2 \log m_2 < 1.7628 m_1 m_2.$$

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If $x^2 + 1 = m_1^{e_1} m_2^{e_2}$ or $2m_1^{e_1} m_2^{e_2}$ with $m_1 \in \{5, 13, 17, 37, 41, 65\}$, then, for $m_1 m_2 \geq 10^{10}$, we have

$$e_1 \log m_1 + e_2 \log m_2 \leq A_4(\log m_1 \log m_2) \log^2(A_5 \log m_1)$$

and otherwise

$$e_1 \log m_1 + e_2 \log m_2 < 1.7628 m_1 m_2.$$

So that, we are led to exponential diophantine equations

$$x^2 + 1 = m_1^{e_1} m_2^{e_2}$$

and

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If $x^2+1=m_1^{e_1}m_2^{e_2}$ or $2m_1^{e_1}m_2^{e_2}$ with $m_1 \in \{5, 13, 17, 37, 41, 65\}$ and $m_1 m_2 > 10^{10}$, then $e_1 \log m_1 + e_2 \log m_2 < C_1 \log m_2$, where C_1 is given in the table below.

Table: Constants used in the proof

m_1	C_1	C_2
5	5586652	2.24×10^{18}
13	7701563	2.69×10^{18}
17	4393299	7.5×10^{17}
37	4200431	5.33×10^{17}
41	9731735	3.01×10^{18}
65	8529020	2.43×10^{18}

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Lemma 3

Moreover, in the case $m_1 = 5$, if $m_2 > 2 \times 10^7$, then, using the same argument as in Y. 2018/2025, we can prove that

$$e_1 \log m_1 + e_2 \log m_2 < A_4 \log 5 \log m_2 \log^2(A_5 \log 5),$$

where $A_4 = 8443.506$ and $A_5 = 966682877$. Hence,

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If $x_1 \in \{2,4,5,6,8,9\}$, then we can take $m_1 = x^2+1$ or $(x^2+1)/2$ and $\eta_1 = x_1 + \sqrt{-1}$. Thus, Størmer's criteria gives that

$$x_2^2+1 = 2^s m_1^a m_2^b, x_3^2+1 = 2^t m_1^c m_2^d$$

with $s,t \in \{0,1\}$ and

$$\left[\frac{x_2+\sqrt{-1}}{x_2-\sqrt{-1}}\right]^d \left[\frac{x_3+\sqrt{-1}}{x_3-\sqrt{-1}}\right]^{\pm b} = \left[\frac{x_1+\sqrt{-1}}{x_1-\sqrt{-1}}\right]^{\pm ad \pm bc}.$$

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Upper bounds for k_1

Hence, we have (7) with

$$g(k_1, k_2, k_3) = (\pm ad \pm bc, d, \pm b), \tag{8}$$

g a nonzero integer (since we have assumed that $\gcd(k_1, k_2, k_3) = 1$), an appropriate combination of signs.

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Case I. $m_1 m_2$: large

For simplicity, we limit ourselves to the case $x_1 = 2$. If $m_1 m_2 > 10^8$, then: Lemma 3 yields that

$$a \log m_1 + b \log m_2, c \log m_1 + d \log m_2 < 6087578 \log m_2.$$

We see that

$$|k_1| \leq ad + bc < \frac{(6087578 \log m_2)^2}{\log 5 \log m_2} = \frac{6087578^2 \log m_2}{\log 5}.$$

Since $\log m_2 < 115315$ by Lemma 1, we obtain

$$|k_1| < 3.71 \times 10^{13} \log m_2 < 4.27 \times 10^{18}, |k_2|, |k_3| \leq 6087577.$$

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Using the simple continued fraction of $4\arctan(1/2)/\pi$, we see that

$$|\delta| := \left|k_2 \arctan\frac{1}{x_2} + k_3 \arctan\frac{1}{x_3}\right| = \left|k_1 \arctan\frac{1}{2} - \frac{r\pi}{4}\right| > 1.303\times 10^{-20},$$

which yields that $x_2 < 7.68 \times 10^{19}(|b| + |d|) < 9.36 \times 10^{26}$ and $m_2 \leq x_2^2 + 1 < 8.77 \times 10^{53}$.

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Hence, we obtain $a \leq \left\lfloor \log(x^2+1)/\log 5 \right\rfloor \leq 77$ and $b \leq \left\lfloor \log(8.77 \times 10^{53})/\log m_2 \right\rfloor \leq 7$.

Now (8) gives $|k_1| \leq 938558423$ and

$$|\delta| > 5.83 \times 10^{-11}.$$

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Iterating our argument with the aid of $b \leq 2$, we have $x_2 < 1.05 \times 10^{17}$, $m_2 \leq 1.11 \times 10^{34}$, and $a \leq 48$.

Hence, $|k_1| \leq 588716504$.

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Case II. $m_1 m_2$: small

Similarly, if $m_1 m_2 \leq 10^8$ and $m_2 > 100$, then

Lemma 2 yields that

$$a \log m_1 + b \log m_2, c \log m_1 + d \log m_2 < 1.7628 \times 10^8.$$

We see that

$$|k_1| \leq ad + bc < \frac{(1.7628 \times 10^8)^2}{\log m_1 \log m_2} < 4.2 \times 10^{15}, |k_2|, |k_3| \leq 38278715.$$

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Using the simple continued fraction of $4\arctan(1/2)/\pi$, we obtain

$$|\delta| > 4.3 \times 10^{-15},$$

which yields that $x_2 < 7.45 \times 10^{23}$ and $x_2^2 + 1 < 5.56 \times 10^{47}$.

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Like above, we obtain $a \leq 68$, $b \leq 23$, $|k_1| \leq 1112253745$, and

$$|\delta| > 5.83 \times 10^{-11},$$

which yields that $x_2 < 6.57 \times 10^{17}$ and $x_2^2 + 1 < 4.32 \times 10^{35}$. Hence, we obtain $a \leq 50$, $b \leq 17$, and

$$|k_1| \leq 819932216.$$

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$$|k_1| \le 819932216.$$

Checking each k_1

For each $k_1 \leq 819932216$, we examined the simple continued fraction of δ to see that

$$\left| k_3 \arctan \frac{1}{x_3} \right| = \left| \delta + k_2 \arctan \frac{1}{x_2} \right| > 6.456 \times 10^{-42}.$$

Since $b \leq 2$, we have

$$x_3 < 3.098 \times 10^{41}, m_3 < 9.6 \times 10^{82},$$

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We can confirm that the equation $x^2 + 1 = 2^s 5^a y^b$ with $3 \le b \le 7$ and $x, y > 1$, which can be reduced to Thue equations with degree ≤ 7 by factoring in Gaussian integers, has no integer solution except $(x, y, s, a, b) = (239, 13, 1, 0, 4)$.

Hence, we must have $b \le 2$.

Thus, we see that $|k_1| \le 390$.

For any (k_1, r) with $|k_1| \le 390$, we must have $|\delta| > 0.00062$ and $x_2 \le 6451$.

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For any (k_1, r) with $|k_1| \le 390$, we must have $|\delta| > 0.00062$ and $x_2 \le 6451$.

Examining each $x_2 \leq 6451$, we found no new solution to (7). This proves Theorem 1.

Addendum (Mar 31. 2025)

We would like to provide a detail on the argument checking each k_1 to conclude $x_3 < 3.098 \times 10^{41}$.

We begin by confirm that:

Lemma 4

If we have $x^2 + 1 = 2^s 5^a m^b$ for some nonnegative integers $x, s, a, b,$ and m with $1.4 \times 10^9 < x < 10^{30}$, then $m^b > 2 \times 10^9$ and $b \leq 2$.

Indeed, if $1.4 \times 10^9 < x < 10^{30}$ and $m^b \leq 2 \times 10^9$, then $13 \leq a \leq 85$.
For each a, we confirmed that $5^a \mid (x^2 + 1)$ implies
$(x^2 + 1)/5^a > 4 \times 10^9$.
Now we must have $b \leq 6$ and then $b \leq 2$ like above.

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Now we must have $b \leq 6$ and then $b \leq 2$ like above.

Put

$$\delta_0 = k_1 \arctan \frac{1}{2} + k_2 \arctan \frac{1}{x_2} - \frac{r\pi}{4}.$$

In the case $b = 1$, we obtain

Lemma 5

If $b = 1$ and $|\delta_0| < 10^{-35}$, then $|\delta| < 1/8$ or $x_3 < 3.098 \times 10^{41}$.

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$$\delta_0 = k_1 \arctan \frac{1}{2} + k_2 \arctan \frac{1}{x_2} - \frac{r\pi}{4}.$$

In the case $b = 1$, we obtain

Lemma 5

If $b = 1$ and $|\delta_0| < 10^{-35}$, then $|\delta| < 1/8$ or $x_3 < 3.098 \times 10^{41}$.

We checked each $x_2 \leq 50000000$ and each $x_2 > 50000000$ with $x_2^2 + 1 = 2^s 5^a m_2$ and $m_2 \leq 10^9$ (we can confirm that there exist only a limit number of such x_2's in a similar way to confirm Lemma 4) to see that $|\delta_0| \geq 10^{-35}$ for such x_2.

Hence, we must have $m_2 > 2 \times 10^7$, which limits us to Case I, and $x_2 > 50000000$.

Thus, we see that

$$|\delta| < 0.122 + \frac{6087577}{x_3}$$

and the Lemma holds.

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and the Lemma holds.

Lemma 6

If $b = 1$ and $|\delta| < 1/8$, then $x_3 < 3.098 \times 10^{41}$.

Put

$$\delta_1 = \delta + \frac{k_2}{x_2} = k_1 \arctan \frac{1}{2} - \frac{r\pi}{4} + \frac{k_2}{x_2}.$$

If $|\delta_1| \geq 1/(2x_2^2)$, then, noting that $|k_2/x_2| < 1/8$ and $x_2 < 1.05 \times 10^{17}$,

$$\begin{aligned} |\delta_0| &= \left| k_1 \arctan \frac{1}{2} + k_2 \arctan \frac{1}{x_2} - \frac{r\pi}{4} \right| \\ &> |\delta_1| - \frac{|k_2|}{3x_2^3} > \frac{11}{24x_2^2} > 10^{-40}. \end{aligned}$$

Lemma 6

If $b = 1$ and $|\delta| < 1/8$, then $x_3 < 3.098 \times 10^{41}$.

Put

$$\delta_1 = \delta + \frac{k_2}{x_2} = k_1 \arctan\frac{1}{2} - \frac{r\pi}{4} + \frac{k_2}{x_2}.$$

If $|\delta_1| \geq 1/(2x_2^2)$, then, noting that $|k_2/x_2| < 1/8$ and $x_2 < 1.05 \times 10^{17}$,

$$\begin{aligned}|\delta_0| &= \left|k_1 \arctan\frac{1}{2} + k_2 \arctan\frac{1}{x_2} - \frac{r\pi}{4}\right| \\ &> |\delta_1| - \frac{|k_2|}{3x_2^3} > \frac{11}{24x_2^2} > 10^{-40}.\end{aligned}$$

Lemma 6

If $b = 1$ and $|\delta| < 1/8$, then $x_3 < 3.098 \times 10^{41}$.

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If $|\delta_1| < 1/(2x_2^2)$, then k_2/x_2 must be a convergent to $-\delta$.
Checking each convergent to $-\delta$ (we note that since r is uniquely determined for each k_1 since $|\delta| < 1/8$), we have

$$\left| k_3 \arctan \frac{1}{x_3} \right| = \left| \delta + k_2 \arctan \frac{1}{x_2} \right| > 6.456 \times 10^{-42}$$

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In the case $b = 2$, we have $x_2^2 + 1 = 2y^2$, $5y^2$, or $10y^2$ and $x_2 < 7.45 \times 10^{23}$.

We check each x_2 to obtain no new solution to (7).

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For other x_1, we have $m_1 \in \{5, 13, 17, 37, 41, 65\}$ and $|k_1| < C_2$, where C_2 is given in Table 2, and a similar argument gives Theorem 2.

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Related problems

(I) Find all integer solutions (x, a, b, q) of $x^2 + 1 = 2^s p^a q^b$ with $s \in \{0, 1\}$, $p \in \{5, 13, 17, 37, 41\}$, $a \geq 0$, $b \geq 3$, and q a prime.

(II) Find all integer solutions of (x, y, p) of $x^2 + 1 = Dy^{2p}$ with $D \in \{5, 10, 17, 26, 37, 65, 82\}$.

(I) with q an integer (not necessarily a prime) would yield the complete solution of (7) with $2 \leq x_1 \leq 7$ or $x_1 = 9$ but seems to be extremely difficult. The limited problem (II) would be more accesible (but a problem with n an integer (possibly odd) in place of $2p$ would be more hard).

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For example, in the case $D = 10$, we assume that $p \geq 11$ and put

$$V_m = \omega^m + \overline{\omega}^m, U_m = \frac{\omega^m - \overline{\omega}^m}{\omega - \overline{\omega}}$$

with $\omega = 3 + \sqrt{10}$ and we see that

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Then, Bennett-Skinner method applied to the Frey curve

$$E_n : Y^2 = X^3 + 2V_{2n+1}X^2 + 10y^{2p}X$$

yields that the Galois representation ρ_n^E associated to E_n arises from a cuspidal newform of weight 2, level 640, and trivial Nebentypes character.

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There are twelve such newforms 640.2.a.a–640.2.a.l according to LMFDB.

640.2.a.i-640.2.a.l: defined over $\mathbb{Q}(\sqrt{5})$. In this case, we have a contradiction examining c_ℓ and $a_\ell(E_n)$ for a few primes ℓ.

640.2.a.d: Taking $\ell = 13$ and observing that $c_{13} = -2$ while $a_{13}(E_n) \in \{2, -6\}$, we must have $p \mid 48$, which is a contradiction.

Similarly, we see that $\rho_p(E_n)$ cannot arise from 640.2.a.a–d or 640.2.a.g–h.

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The 640.2.a.f case

Bugeaud-Mignotte-Siksek would allow us to settle 640.2.a.f but it would require a considerable amount of computation.

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MANY THANKS FOR YOUR ATTENTION

To be continued...?

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