Integers whose sum of divisors is a prime power

Tomohiro Yamada (CJLC, Osaka University)

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Introduction

As usual, let

- $\sigma(N)$: the sum of divisors of N,
- $\omega(n)$: the number of distinct prime factors of n,
- $\tau(n)$: the number of divisors of n.

We study arithmetic properties of integers N satisfying

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Factoring $N = \prod_i q_i^{f_i}$ with $q_1 < q_2 < \cdots$ primes, (1) immediately yields that

$$\frac{q_i^{s_i} - 1}{q_i - 1} = p^{g_i} \tag{2}$$

for each i, where $s_i = f_i + 1$ and g_i is a certain positive integer.

It is conjectured that

- only $(q_i, s_i) = (3, 5)$ satisfies (2) with $g_i \geq 2$ and
- (2) with $g_i = 1$ has at most one solution except $2^5 - 1 = 5^2 + 5 + 1 = 31$

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Main results

Theorem 1 (Y.)

If $\sigma(N) = p^e$ for some prime p, then $\sigma(N)/N < 5.388$.

Theorem 2 (Y.)

If $\sigma(N) = p^e$ for some prime $p \geq \exp\exp x_i$, then $q_2 > \exp c_i$, where $(x_i, c_i) = (42.04, 14), (42.31, 15), (42.56, 16), \ldots, (46.67, 45), \ldots$.

Theorem 3 (Y.)

If $\sigma(N) = p^e$ for some prime $p \geq \exp\exp 46.67$, then $\sigma(N)/N < 1.1$ or $p = \sigma(q_1^{f_1})$ with $q_1 \leq 31$.

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Background

An integer N is called perfect if

$$\sigma(N) = 2N.$$

It is well known that an integer N is even perfect if and only if $N = 2^{p-1}(2^p - 1)$ with $2^p - 1$ prime.

Unsolved problem

Is there any odd perfect number?

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- A divisor d of N is called a unitary divisor if $\gcd(d, N/d) = 1$.
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- More generally, we can define k-ary divisor recursively. A divisor d of N is called a $(k+1)$-ary divisor if $\gcd_k(d, N/d) = 1$.
- Moreover, it is known that if p^m is a $(e-1)$-ary divisor of p^e, then p^m is a k-ary divisor of p^e for any $k \geq e-1$. A divisor $d = \prod_i p_i^{m_i}$ of $N = \prod_i p_i^{e_i}$ is called an infinitary divisor if each $p_i^{m_i}$ is an infinitary divisor of $p_i^{e_i}$.

The sum of k-ary divisors of N is denoted by $\sigma^{(k)}(N)$ or $\sigma^{**\cdots*}(N)$ with k stars.
We note that $\sigma^*(N) = \prod_i (p_i^{e_i} + 1)$ and, writing $e_i = 2^{k_{i,1}} + \cdots + 2^{k_{i,t_i}}$ with $k_{i,1} > \cdots > k_{i,t_i}$ for each i, $\sigma^{(\infty)}(N) = \prod_{i,j} (p_i^{2^{k_{i,j}}} + 1)$.

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Are there only finitely many even k-ary perfect numbers for each k?

The only settled case is $k = 2$: $6, 60, 90$ are the only biunitary perfect numbers (Wall, 1971).

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Y., 2017

9 is the only odd infinitary superperfect number.

Y., 2018

2 and 9 are the only biunitary superperfect numbers, EVEN or ODD!

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Our motivation

Hybrid two $\sigma^{(k)}$ functions!

- $\sigma^*(\sigma(N)) = 2N$: $27, 63, 165, 238$, and 2^{p-1} with $2^p - 1$ prime
- $\sigma^{**}(\sigma(N)) = 2N$: 2^{p-1} with $2^p - 1$ prime
- $\sigma(\sigma^*(N)) = 2N, \sigma(\sigma^{**}(N)) = 2N$: $2, 9$

c.f.

- $\sigma^*(\sigma(N)) = kN$: $1, 2, 4, 8, 10, 16, 24, 27, 30, 54, 63, 64, 108, 126, \ldots$ (OEIS: A045795)
- $\sigma(\sigma^*(N)) = kN$: $2, 9, 15, 18, 21, 40, 42, 60, 104, 120, 288, 312, 756, \ldots$ (OEIS: A083288)
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- $\sigma^*(\sigma(N)) = kN$: $1, 2, 4, 8, 10, 16, 24, 27, 30, 54, 63, 64, 108, 126, \ldots$ (OEIS: A045795)
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If $\sigma^{(k)}(\sigma(N)) = 2N$ for some $k \in \{1, 2, \ldots, \infty\}$ and N is odd, then

$$\sigma(N) = 2^s p^t \tag{3}$$

for an odd prime p and integers s, t.

With the aid of Bang-Zsigmondy theorem, Størmer's theorem, and Ljunggren's theorem on $Y^2 + 1 = 2X^4$ (for a relatively simple proof, see Steiner and Tzanakis, 1991), we see that q_i satisfies (2) except when

- $(q_i, f_i) = (2^{h_i} p^{g_i} - 1, 1)$ or
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Related results

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For a fixed N,

$$\frac{x^m - 1}{x - 1} = N$$

has at most $\log^{1/4+o(1)} N$ solution pairs (x, m).

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For each r, there exist only finitely many odd superperfect numbers N with $\omega(N) \le r$ or $\omega(\sigma(N)) \le r$.

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These results are based on the following idea: if values of the form $\frac{x_i^{m_i}-1}{x_i-1}$ are multiplicatively dependent, then, using Baker's method, we can show that x_i's cannot be all small.

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Ljunggren's conjecture

The equation

$$\frac{x^k - 1}{x - 1} = y^\ell \qquad (4)$$

with $x, y, \ell \geq 2$ and $k \geq 3$ has no solution other than

$$\frac{3^5 - 1}{3 - 1} = 11^2, \frac{18^3 - 1}{18 - 1} = 7^3, \frac{7^4 - 1}{7 - 1} = 20^2. \qquad (5)$$

We use the following result.

Bugeaud and Mignotte, 2002

(4) has no solution other than (5) in the range $x \leq 10^6$.

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Aleksentsev, 2008 applied to the rational case (settings)

Let

- $a_1, \ldots, a_n$: rational integers ≥ 2,
- $b_1, \ldots, b_n$: rational integers,

and

$$\Lambda = b_1 \log a_1 + \cdots + b_n \log a_n.$$

Sort a_i's and b_i's in such a way that

$$|b_n| \log^+ a_n = \max_{1 \leq i \leq n} |b_i| \log^+ a_i$$

Moreover, we write $\log^+ x = \max\{1, \log x\}$.

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Moreover, we write $\log^+ x = \max\{1, \log x\}$.

Aleksentsev, 2008 applied to the rational case

Then $\Lambda = 0$ or

$$\log|\Lambda| > -C(n)\log^+ a_1 \cdots \log^+ a_n \log B,$$

where

$$B = \max\left\{3, \max_{1 \le i \le n-1} \frac{|b_i|}{\log^+ a_n} + \frac{|b_n|}{\log^+ a_i}\right\}$$

and

$$C(n) = 5.3n(n+1)(n+5)(n+8)^2 31.44^n \frac{(n+1)^n}{n^{n+0.5}} \log(3n).$$

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We also use the following classical result.

Bang's theorem

If a and n are integers ≥ 2, then $a^n - 1$ has a prime factor dividing none of $a^m - 1$ with $1 \leq m \leq n-1$ unless $(a, n) = (2, 6)$ or $(a, n) = (2^k - 1, 2)$ for some integer $k \geq 2$.

Now we see that for each divisor $d > 2$ of s_i, $q_i^d - 1$ has a prime factor dividing none of $q_i^m - 1$ with $1 \leq m \leq d-1$. Hence, we obtain

Corollary

$\sigma(q_i^{s_i - 1})$ has at least $\tau(s_i) - 1$ distinct prime factors for any odd prime q_i.

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Diophantine Analysis

We begin with

$$p^{g_i} = \frac{q_i^{s_i} - 1}{q_i - 1}.$$

From the above Corollary of Bang's theorem, we see that s_i must be a prime factor of $p - 1$.

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We set

$$T_k = \{q_i : s_i = k\} = \{q_i : f_i = k - 1\}.$$

We note that k must be a prime factor of $p - 1$ whenever T_k is not empty.
Hence,

$$\frac{q_i^k - 1}{q_i - 1} = p^{g_i}$$

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Lemma 1

If $q_i < q_j$ both belong to T_k, then $q_j > q_i^{k-1}$.

Since $q_j > q_i$,

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Since every $X = q_i, q_i^2, \ldots, q_i^{k-1}$ satisfies

$$\frac{X^k - 1}{X - 1} \equiv 0 \pmod{p^{g_i}} \tag{8}$$

and

$$q_i < q_i^2 < \cdots < q_i^{k-1} < \frac{q_i^k - 1}{q_i - 1} = p^{g_i},$$

$X = q_i, q_i^2, \ldots, q_i^{k-1}$ give all solutions to (8) with $0 \le X < p^{g_i}$.
Hence, $q_j \equiv q_i^t \pmod{p^{g_i}}$ for some t with $1 \le t \le k-1$.
We can easily see that $q_j \neq q_i^t$ and therefore $q_j > p^{g_i} > q_i^{k-1}$.

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Lemma 2

For each q_i, we have

$$s_i < 6.33797 \times 10^9 \log p \log q_i (22.4731 + \log\log p)$$

and

$$g_i < 6.33797 \times 10^9 \log^2 q (22.4731 + \log\log p).$$

The three logarithmic form

$$\Lambda_i = s_i \log q_i - \log(q_i - 1) - g_i \log p$$

satisfies

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By Aleksentsev's bound, we have

$$\log(q_i^{s_i} - 1) = -\log \Lambda_i < C_0(3) \log q_i \log(q_i - 1) \log p \log B_0,$$

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Lemma 3

$$\log p < \log(1.2676 \times 10^9 (22.4731 + \log\log p) \log^2 q_2) + C(4) \log^2 q_1 \log^2 q_2 \log B_1,$$

where

$$B_1 = 8.13571 \times 10^{19} \log^2 q_2 \log p (22.4731 + \log\log p)^2.$$

Clearly we have

$$\left(\frac{q_2^{s_2} - 1}{q_2 - 1}\right)^{g_1} = \left(\frac{q_1^{s_1} - 1}{q_1 - 1}\right)^{g_2}.$$

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We see that

$$g_1\left(s_2\log q_2-\log(q_2-1)-\log\frac{q_2^{s_2}}{q_2^{s_2}-1}\right)$$
$$=g_2\left(s_1\log q_1-\log(q_1-1)-\log\frac{q_1^{s_1}}{q_1^{s_1}-1}\right)$$

and, putting

$$\Lambda=g_1s_2\log q_2-g_1\log(q_2-1)-g_2s_1\log q_1+g_2\log(q_1-1),$$

we have

$$0<|\Lambda|<\frac{g_1}{q_2^{s_2}-1}+\frac{g_2}{q_1^{s_1}-1}<\frac{g_1+g_2}{p},$$

where we note that $\Lambda\neq 0$ since $g_1s_2\neq 0$.

We see that

$$g_1 \left(s_2 \log q_2 - \log(q_2 - 1) - \log \frac{q_2^{s_2}}{q_2^{s_2} - 1} \right)$$
$$= g_2 \left(s_1 \log q_1 - \log(q_1 - 1) - \log \frac{q_1^{s_1}}{q_1^{s_1} - 1} \right)$$

and, putting

$$\Lambda = g_1 s_2 \log q_2 - g_1 \log(q_2 - 1) - g_2 s_1 \log q_1 + g_2 \log(q_1 - 1),$$

we have

$$0 < |\Lambda| < \frac{g_1}{q_2^{s_2} - 1} + \frac{g_2}{q_1^{s_1} - 1} < \frac{g_1 + g_2}{p},$$

where we note that $\Lambda \neq 0$ since $g_1 s_2 \neq 0$.

We see that

$$g_1\left(s_2\log q_2-\log(q_2-1)-\log\frac{q_2^{s_2}}{q_2^{s_2}-1}\right)$$
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From Lemma 2, we have

$$g_i < 6.33797 \times 10^9 \log^2 q_i (22.4731 + \log\log p)$$

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$$0 < |\Lambda| < \frac{1.2676 \times 10^9 (22.4731 + \log\log p) \log^2 q_2}{p}.$$

Now we obtain

$$-\log|\Lambda| < C(4) \log q_1 \log(q_1 - 1) \log q_2 \log(q_2 - 1) \log B_1.$$

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Proof of main results

We begin by Theorem 1. If $p = 2$, then $q_i^{f_i}$ must be a Mersenne prime and

$$\frac{\sigma(N)}{N} < \prod_{2^\ell - 1:\text{prime}} \frac{2^\ell}{2^\ell - 1} = 1.58555888\cdots.$$

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If $p \geq \exp\exp 41.3$, then, using Lemma 3,

$$q_2 > \log^{0.2787} p$$

and, using Rosser-Schoenfeld inequality, we have

$$\prod_{i \geq 2, q_i < \log p} \frac{q_i}{q_i - 1} < 3.61,$$

where we note recent estimates for functions over primes would give a slightly better estimate.

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Now our concern is to q_i's above $\log p$. For each prime factor k of $p-1$, we write p_k for the smallest $q_i > \log p$ in T_k if it exists. Then, using Lemma 1, we have

$$\prod_{q_i > \log p, q_i \in T_k} \frac{q_i}{q_i - 1} \leq \prod_j \frac{p_k^{(k-1)^j}}{p_k^{(k-1)^j} - 1} < 1 + \frac{1 + 10^{-8}}{p_k}.$$

We note that $\omega(p-1) < 1.38402 \log p / \log\log p$ (Robin, 1983) and

$$\prod_{q_i > \log p} \frac{q_i}{q_i - 1} < \left(1 + \frac{1}{\log p}\right)^{1.38402 \log p / \log\log p} < 1.035.$$

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For $2 < p < \exp\exp 41.3$, we use BGM2020 and BM2002 to obtain $q_2 > 10^5$. Rosse-Schoenfeld inequality gives

$$\begin{aligned}\prod_{i\geq 2, q_i<\log p} \frac{q_i}{q_i-1} &< \prod_{10^5<q<\log p} \frac{q}{q-1}\\ &< \frac{\log\log p}{10\log 5}\left(1+\frac{1}{2\log^2(10^5)}\right)\left(1+\frac{1}{2(\log\log p)^2}\right)\end{aligned}$$

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In both cases, if $q_1 \geq 17$, then

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Assume that $q_1 = 3$.

- If $p \geq \exp\exp 39.75$, then Lemma 3 gives $q_2 > \log^{0.28963} p$ and therefore

$$\prod_{i \geq 2, q_i < \log p} \frac{q_i}{q_i - 1} < 3.467$$

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- If $p < \exp\exp 39.75$, then, like above,

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Theorem 2 is a straightforward application of Lemma 3.

For Theorem 3, we proceed like in the proof of Theorem 1 with $q_2 \geq e^{45}$ from Theorem 2 to obtain

$$\prod_{i \geq 2} \frac{q_i}{q_i - 1} < 1.0686.$$

If $q_1 \geq 37$, then $\sigma(N)/N < 1.0686 \times 37/36 < 1.1$, proving Theorem 3. We note that if $q_1 \leq 31$ and $p \geq \exp\exp 46.7$, then we must have $\sigma(q_1^{f_1}) = p$ from BM2002.

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Unsolved problems

(I) For every integer $k \geq 0$, are there only finitely many integers N satisfying $\sigma^{(k)}(\sigma(N)) = 2N$ not of the form 2^{p-1}?

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MANY THANKS FOR YOUR ATTENTION

To be continued...?

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